Lecture 21

11.1 - Sequences

Def: A sequence is a list of numbers written in a particular order:

$$\{a_1, a_2, a_3, \cdots\}$$

A sequence can be a finite or infinite list. The n^{th} term of the sequence is the n^{th} number in the list.

Ex: In the sequence $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$, what is the 2^{nd} term? the 6^{th} term? the n^{th} term?

There are several ways to represent a sequence. Common ones are: (for this example)

$$\{-1, \frac{1}{4}, \frac{-1}{9}, \frac{1}{16}, \cdots\}, \quad a_n = \frac{(-1)^n}{n^2}, \quad \left\{\frac{(-1)^n}{n^2}\right\}, \quad \left\{\frac{(-1)^n}{n^2}\right\}_{n=1}^{\infty}$$

(Note: There may not always be a formula for the n^{th} term!)

Ex: Find a formula for the n^{th} term of the sequence: $\left\{1, -\frac{1}{4}, \frac{1}{16}, -\frac{1}{64}, \frac{1}{256}, \cdots\right\}$

Graphing a Sequence

To graph a sequence $\{a_1, a_2, a_3, \cdots\}$, we think of it as a function from positive integers to real numbers, and plot the points $(1, a_1), (2, a_2), (3, a_3), \cdots, (n, a_n), \cdots$

What we get is a plot of dots, one over each positive integer.

See mathematica code for examples.

Ex 1: $a_n = \frac{n+1}{n}$ **Ex 2:** $a_n = \frac{\cos(\frac{n}{4})}{n}$ **Ex 3:** $a_n = n \sin \frac{n}{4}$

Limit of a Sequence

Notice in examples 1 & 2 that the points start clustering towards a specific value, or getting nearer to a horizontal line/asymptote. This is because these sequences are *converging* to a value L and this line is $y=L$. Example 3 is an example of a *divergent* sequence.

Def: A sequence $\{a_n\}$ has the *limit* L and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \longrightarrow L \text{ as } n\to\infty$$

if we can make the terms of a_n as close to L as we like by taking n sufficiently large. More precisely, if $\forall \varepsilon > 0$ $\exists$ integer $N>0$ such that for $n>N$, $|a_n - L| < \varepsilon$. A sequence is *convergent* if it has a limit, and *divergent* otherwise.

Theorem:

If there is a function $f(x)$ such that $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ for integers n, then

Ex: Show that the limit of $a_n = \frac{n+1}{n}$ is indeed 1.

By this theorem, we can use all of our usual techniques for limits to find limits of sequences, such as L'Hôpital's rule:

Ex: Is the sequence $\left\{\frac{n^2}{4^n}\right\}_{n=1}^{\infty}$ convergent?

There are two ways a sequence can diverge: it can diverge by not approaching any value (as in Example 3), or it can <u>diverge to infinity</u>, e.g., $a_n = 2^n$.

All the usual rules for limits apply to sequences too:

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences, and c is any constant:

ⓐ $$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$$

ⓑ $$\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n$$

ⓒ $$\lim_{n\to\infty} (a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$$

ⓓ $$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$$ (as long as $\lim_{n\to\infty} b_n \neq 0$)

ⓔ $$\lim_{n\to\infty} a_n^p = \left(\lim_{n\to\infty} a_n\right)^p$$ if $p>0$ and $a_n > 0$.

ⓕ If $\lim_{n\to\infty} a_n = L$ & $f(x)$ is continuous

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L)$$

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Ex: Is the sequence convergent or divergent?

$$\left\{ \tan\left(\frac{2n\pi}{1+8n}\right) - \frac{n}{1+n^2} \right\}_{n=1}^{\infty}$$

If it converges, what is the limit?

The Squeeze Theorem

If $a_n \le b_n \le c_n$ for $n \geq n_0$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then

Ex: Find the limit of $a_n = \dfrac{\cos\left(\frac{n}{4}\right)}{n}$.

Alternating Sequences

For any sequence $\{a_n\}$, we have

$$-|a_n| \leq a_n \leq |a_n|$$

so, by the squeeze theorem

In fact, a sequence with infinitely many positive & negative terms converges if and only if $\lim_{n\to\infty} |a_n| = 0$. In any other case, the limit does not exist.

Ex: Do the sequences

$$\left\{(-1)^n \frac{2n+1}{n^2}\right\}_{n=1}^{\infty} \quad \& \quad \left\{(-1)^n \frac{2n+1}{n}\right\}_{n=1}^{\infty}$$

converge?

Monotone Sequences

<u>Def</u>: A sequence $\{a_n\}$ is called:

- <u>increasing</u> if $a_n < a_{n+1}$ for all $n \geq 1$
- <u>decreasing</u> if $a_n > a_{n+1}$ for all $n \geq 1$

A <u>monotone sequence</u> is one which is increasing or decreasing.

<u>Def</u>: A sequence $\{a_n\}$ is:

- <u>bounded above</u> if $\exists M$ such that $a_n \leq M$ for $n \geq 1$
- <u>bounded below</u> if $\exists m$ such that $a_n \geq m$ for $n \geq 1$

A sequence is <u>bounded</u> if it is both bounded above and below.

<u>Theorem</u>: Every bounded monotonic sequence is convergent.

<u>Check for Monotonicity</u>: If our sequence can be written $a_n = f(n)$, where f is differentiable, $\{a_n\}$ is

- <u>increasing</u> if $f'(x) > 0$
- decreasing if $f'(x) < 0$

Def: Is the sequence $\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \cdots\}$ monotone? bounded? convergent?